CONVERGENCE OF THE ZERO-CROSSING RATE OF
AUTOREGRESSIVE PROCESSES AND ITS LINK
TO UNIT ROOTS

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ABSTRACT

The asymptotic zero-crossing rate (ZCR) of the general first and second order autoregressive processes is investigated. When the associated characteristic polynomial has a unit root $e^{i\theta}$, $0 \leq \theta \leq \pi$, the ZCR converges in mean square to θ/π, and the rate of convergence is very fast regardless of the noise level. It is conjectured that in higher order autoregressive processes multiple unit roots can be determined by the ZCR of the filtered processes. An indication to this effect is given.

Convergence of the Zero-Crossing Rate of Autoregressive Processes and its Link to Unit Roots

1. Introduction

In this paper we investigate the convergence of the zero-crossing rate (ZCR) of nonstationary first and second order autoregressive processes. In addition to other parameters, of special concern to us is the link between the unit roots of AR(2) processes and their ZCR. Our main result states that the ZCR of a nonstationary AR(2) whose characteristic equation has a unit root $e^{i\theta}$, $0 \le \theta \le \pi$, converges to θ/π in quadratic mean. It seems that the convergence is very fast and is essentially identical to the case of a pure sinusoid regardless of the magnitude of the additive noise component. At present it is not clear how to extend this result to the general nonstationary AR(p) with unit roots, but it is *conjectured* that the ZCR of the filtered processes may be the required remedy. An indication of this fact will be illustrated at the end.

Let $\{X_n\}$ be a Gaussian process with zero mean and where $n = 0,\pm1,\pm2,\ldots$, and let $\{\chi_n\}$ be the corresponding clipped process

$$\chi_n = \begin{cases} 1, & X_n \ge 0 \\ 0, & X_n < 0 \end{cases}, \qquad n = 0,\pm1,\pm2,\ldots .$$

The quantity

$$\mathcal{D}_N(\chi) = \sum_{j=1}^{N} (\chi_j - \chi_{j-1})^2$$

is called the number of zero-crossings of the time series $X_0, X_1, X_2, \ldots, X_N$. $\mathcal{D}_N(\chi)/N$ is the corresponding ZCR. When $\{X_n\}$ follows a first order autoregressive model

$$X_t = \alpha X_{t-1} + \epsilon_t, \quad \alpha \in R,$$

then under some conditions

$$\underset{N\to\infty}{\mathrm{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathcal{X}) = \begin{cases} \frac{1}{\pi}\cos^{-1}\alpha, & |\alpha| < 1 \\ \frac{1}{\pi}\cos^{-1}\frac{\alpha}{|\alpha|}, & |\alpha| \geq 1 . \end{cases}$$

The complete statement is given in Theorem 2.1.

Let B denote the backward shift operator, and consider the nonstationary AR(2) process with two unit roots

$$(1 - e^{i\theta}B)(1 - e^{-i\theta}B)X_t = \epsilon_t, \quad 0 \leq \theta \leq \pi.$$

Then under appropriate conditions

$$\underset{N\to\infty}{\mathrm{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathcal{X}) = \frac{\theta}{\pi}.$$

A more complete statement is given in Theorem 3.1.

The paper is organized as follows. After some preliminaries, section 2 treats the first order case while section 3 deals with the second order case. The method of proof is the same in both cases. Section 4 deals with unit roots.

1.1 Preliminaries and motivation

For a better exposition it is helpful to expound on some key observations.

Throughout the paper we assume that $\{X_t\}$, $t = 0,\pm1,\pm2,\ldots$, is a zero mean Gaussian process with correlation function

$$\rho_k(t) = \frac{EX_tX_{t+k}}{(EX_t^2EX_{t+k}^2)^{\frac{1}{2}}} .$$

It turns out that our processes become asymptotically stationary by satisfying the condition

$$\lim_{t\to\infty} \rho_k(t) = \rho_k = \rho_{-k}, \quad k = 0,1,2,\ldots . \tag{1.1}$$

Now, if $\underset{\sim}{a} = (a_1,\ldots,a_k)' \in R^k$, then we have

$$\underset{\sim}{a}' \begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & 1 \end{pmatrix} \underset{\sim}{a} = \lim_{t\to\infty} E\left\{ \sum_{j=1}^{k} \frac{a_jX_{t+j}}{(EX_{t+j}^2)^{\frac{1}{2}}} \right\}^2 \geq 0$$

and therefore the matrix $(\rho_{m_i - m_j})$, $i,j = 1,\ldots,k$, is always nonnegative definite and there exists a stationary Gaussian process $\{Y_t\}$ such that

$$EY_t = 0, \quad EY_tY_{t+k} = \rho_k, \quad k = 0,1,2,\ldots .$$

This fact is fundamental to our development.

Another useful fact is the formula [1, p. 34]

$$Ex_jx_{j+1} = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}\rho_1(j). \tag{1.2}$$

It follows that when (1.1) holds we have

$$\lim_{N\to\infty} \frac{1}{N} E\mathcal{D}_N(\mathcal{X}) = \frac{1}{\pi} \cos^{-1}\rho_1 . \tag{1.3}$$

This is the limiting expected ZCR. It turns out that the ZCR itself achieves this limit in quadratic mean when (1.1) holds.

The motivation for using the ZCR in determining unit roots in non-stationary autoregressive processes springs out from the following fact. A second order autoregressive Gaussian process with two unit roots $e^{\pm i\theta}$ (they are a conjugate pair) has the form

$$X_t = 2\cos\theta\, X_{t-1} - X_{t-2} + \varepsilon_t \tag{1.4}$$

where $\{\varepsilon_t\}$ is Gaussian white noise. Assume for a moment that $\varepsilon_t = 0$ with probability one. Then $\{X_t\}$ is a pure sinusoid with frequency θ and this implies that with probability one

$$\left| \theta - \frac{\pi}{N} \mathcal{D}_N(\mathcal{X}) \right| \leq \frac{\pi}{N},$$

or that

$$\underset{N\to\infty}{\text{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathcal{X}) = \frac{\theta}{\pi}.$$

The surprising fact is that this limit remains true even in the presence of ε_t in (1.4), and in fact <u>regardless</u> of the magnitude of ε_t, that is, regardless of the size of $\mathrm{Var}(\varepsilon_t)$. Moreover, the rate of convergence of the ZCR to θ/π is fast. Evidently, these are robustness properties of the zero-crossing rate.

2. The ZCR of a general AR(1)

Suppose $\{X_t\}$ follows the AR(1) model

$$X_t = \alpha X_{t-1} + \varepsilon_t, \quad t = 0, \pm 1, \ldots, \quad \alpha \neq 0 \tag{2.1}$$

where $\{\varepsilon_t\}$ is Gaussian white noise with mean 0 and variance σ_ε^2. Let $X_0 \equiv \xi$ be an initial value independent of $\{\varepsilon_t\}$ with $E\xi = 0$, $E\xi^2 = \sigma^2$. The solution of (2.1) depends on whether $t \geq 1$ or $t \leq -1$ and is given by

$$X_t = \alpha^t \xi + \alpha^{t-1}\varepsilon_1 + \ldots + \alpha\varepsilon_{t-1} + \varepsilon_t, \quad t \geq 1 \tag{2.2}$$

$$X_t = \alpha^t \xi - \alpha^t \varepsilon_0 - \alpha^{t+1}\varepsilon_{-1} - \ldots - \alpha^{-1}\varepsilon_{t+1}, \quad t \leq -1 . \tag{2.3}$$

Note that $\{X_t\}$ is not a stationary process but that (2.1) has a stationary solution when $|\alpha| \neq 1$. For $s \geq 1$, $t \geq 1$ define

$$\phi_1(t,s) = \frac{(1-\alpha^2)\sigma^2 - \sigma_\varepsilon^2}{(1-\alpha^2)\sigma^2 + (\alpha^{-2(t+s)} - 1)\sigma_\varepsilon^2}$$

$$\psi_1(t,s) = \frac{\alpha^{-2(t+s)}\sigma_\varepsilon^2}{(1-\alpha^2)\sigma^2 + (\alpha^{-2(t+s)} - 1)\sigma_\varepsilon^2} .$$

It follows that uniformly in s and as $t \to \infty$

$$\phi_1(t,s) \to \begin{cases} 0, & |\alpha| < 1 \\ 1, & |\alpha| > 1 \end{cases}$$

and

$$\psi_1(t,s) \to \begin{cases} 1, & |\alpha| < 1 \\ 0, & |\alpha| > 1 . \end{cases}$$

$\rho_s(t)$ can be expressed conveniently by

$$\rho_s(t) = \{\phi_1(t,s) + \alpha^{2s}\psi_1(t,s)\}^{\frac{1}{2}} \cdot (\frac{\alpha}{|\alpha|})^s, \qquad |\alpha| \neq 1$$

$$= \left\{\frac{\sigma^2 + t\sigma_\varepsilon^2}{\sigma^2 + (t+s)\sigma_\varepsilon^2}\right\}^{\frac{1}{2}} \cdot \alpha^s, \qquad |\alpha| = 1 .$$

Similarly, for $t \leq -1$, $s \leq -1$ define

$$\phi_2(t,s) = \frac{\sigma^2(1-\alpha^{-2}) - \sigma_\varepsilon^2\alpha^{-2}}{\sigma^2(1-\alpha^{-2}) + \alpha^{-2}(\alpha^{-2(t+s)} - 1)\sigma_\varepsilon^2}$$

$$\psi_2(t,s) = \frac{\alpha^{-2}\sigma_\varepsilon^2}{\alpha^{2(t+s)}\sigma^2(1-\alpha^{-2}) + \alpha^{-2}(1-\alpha^{2(t+s)})\sigma_\varepsilon^2} .$$

Then again uniformly in s and as $t \to -\infty$

$$\phi_2(t,s) \to \begin{cases} 1, & |\alpha| < 1 \\ 0, & |\alpha| > 1 \end{cases}$$

and

$$\psi_2(t,s) \to \begin{cases} 0, & |\alpha| < 1 \\ 1, & |\alpha| > 1 . \end{cases}$$

The correlation function becomes

$$\rho_s(t) = \{\phi_2(t,s) + \alpha^{2s}\psi_2(t,s)\}^{\frac{1}{2}} \cdot (\frac{\alpha}{|\alpha|})^s, \qquad |\alpha| \neq 1$$

$$= \left\{\frac{\sigma^2 - t\sigma_\varepsilon^2}{\sigma^2 - (t+s)\sigma_\varepsilon^2}\right\}^{\frac{1}{2}} \cdot \alpha^s, \qquad |\alpha| = 1.$$

We therefore have

Lemma 2.1. Assume that $\{X_t\}$ follows the AR(1) model (2.1) where $X_0 \equiv \xi$ is independent of the Gaussian white noise $\{\varepsilon_t\}$. Then

$$\lim_{t\to\infty} \rho_s(t) = \begin{cases} \alpha^s, & |\alpha| < 1 \\ \left(\frac{\alpha}{|\alpha|}\right)^s, & |\alpha| > 1 \end{cases}, \quad \text{uniformly in } s \geq 1.$$

$$\lim_{t\to-\infty} \rho_{-s}(t) = \begin{cases} \left(\frac{\alpha}{|\alpha|}\right)^s, & |\alpha| < 1 \\ \alpha^{-s}, & |\alpha| > 1 \end{cases}, \quad \text{uniformly in } s \geq 1.$$

$$\lim_{t\to\pm\infty} \rho_{\pm s}(t) = \alpha^s, \qquad |\alpha| = 1.$$

Proof. If $|\alpha| < 1$ and $(1-\alpha^2)\sigma^2 = \sigma_\varepsilon^2$ then $\phi_1(t,s) = 0$ and $\psi_1(t,s) = 1$ and so $\rho_s(t) = \alpha^s$ for all t. If $|\alpha| < 1$ and $(1-\alpha^2)\sigma^2 \neq \sigma_\varepsilon^2$ we have

$$|\rho_s(t) - \alpha^s| = |[\phi_1(t,s) + \alpha^{2s}\psi_1(t,s)]^{\frac{1}{2}} - |\alpha|^s|.$$

Because $\phi_1(t,s) \to 0$, $\psi_1(t,s) \to 1$ uniformly in $s \geq 1$ as $t \to \infty$, for any $\varepsilon > 0$ we can find $T_1 > 0$ such that whenever $t \geq T_1$

$$|\phi_1(t,s)| < \frac{\varepsilon^2}{2} \quad \text{and} \quad |\psi_1(t,s)| < 2.$$

Choose s_0 such that $2\alpha^{2s_0} < \frac{\varepsilon^2}{2}$. Then for $t \geq T_1$, $s \geq s_0$

$$|\rho_s(t) - \alpha^s| < \varepsilon.$$

Since $\lim_{t\to\infty} \rho_s(t) = \alpha^s$ for $s = 1,2,\ldots,s_0$, we can find T_0 such that for $t \geq T_0$ we also have $|\rho_s(t) - \alpha^s| < \varepsilon$ for $s = 1,2,\ldots,s_0$. Thus if we put $T = T_1 \vee T_0$ then whenever $t \geq T$

$$|\rho_s(t) - \alpha^2| < \varepsilon \quad \text{for all} \quad s \geq 1.$$

If $|\alpha| > 1$ then obviously $(1-\alpha^2)\sigma^2 \neq \sigma_\varepsilon^2$ and for sufficiently large t

$$|\alpha^{2s}\psi_1(t,s)| = \frac{\alpha^{-2t}\sigma_\varepsilon^2}{|(1-\alpha^2)\sigma^2 + (\alpha^{-2(t+s)} - 1)\sigma_\varepsilon^2|}$$

$$= \frac{\sigma_\varepsilon^2}{\alpha^{2t}[(\alpha^2-1)\sigma^2 + \sigma_\varepsilon^2] - \frac{\sigma_\varepsilon^2}{\alpha^{2s}}}$$

$$\leq \frac{\sigma_\varepsilon^2}{\alpha^{2t}[(\alpha^2-1)\sigma^2 + \sigma_\varepsilon^2] - \frac{\sigma_\varepsilon^2}{\alpha^2}}$$

$$\to 0, \quad t \to \infty \quad \text{uniformly in} \quad s \geq 1.$$

Now

$$|\rho_s(t) - (\tfrac{\alpha}{|\alpha|})^s| = |(\phi_1(t,s) + \alpha^{2s}\psi_1(t,s))^{\frac{1}{2}} - 1|.$$

But since $\phi_1(t,s) \to 1$ uniformly in $s \geq 1$ as $t \to \infty$, and since $|\alpha^{2s}\psi_1(t,s)| \to 0$ also uniformly in $s \geq 1$ as $t \to \infty$, it follows that $\rho_s(t) \to (\frac{\alpha}{|\alpha|})^s$ uniformly in $s \geq 1$ as $t \to \infty$. This takes care of the first part of the lemma. The second part can be proven in the same way, and the third part is obviously true since we do not claim a uniform limit for this case. □

In order to translate Lemma 2.1 into a statement about the convergence of the ZCR we shall use the fact that the process in (2.1) converges in some sense to a stationary process regardless of α. Recall that a sequence of Gaussian processes

$$\mathcal{X}_n = (X_1^{(n)}, X_2^{(n)}, \ldots), \quad EX_i^{(n)} = 0, \quad \text{all} \quad i \geq 1,$$

converges in distribution to a Gaussian process

$$\mathbb{Y} = (Y_1, Y_2, \ldots), \quad EY_i = 0, \quad \text{all} \quad i \geq 1$$

if and only if

$$\lim_{n\to\infty} EX_{m+k}^{(n)} X_m^{(n)} = EY_{m+k} Y_m \quad \text{for all } m,k.$$

We shall say that the convergence is uniform if it is uniform in m and k.

When $|\alpha| < 1$, (2.1) has a stationary solution

$$Y_t = \sum_{j=0}^{\infty} \alpha^j \varepsilon_{t-j}, \quad t = 0, \pm 1, \pm 2, \ldots \tag{2.4}$$

and when $|\alpha| > 1$ (2.1) still has a stationary solution of the form

$$Z_t = -\sum_{j=0}^{\infty} \alpha^{-j+1} \varepsilon_{t+j+1}, \quad t = 0, \pm 1, \ldots . \tag{2.5}$$

Define a sequence of random variables $\{W_i\}$ by

$$W_i = \left(\frac{\alpha}{|\alpha|}\right)^i X, \qquad X \sim N(0,1). \tag{2.6}$$

In accordance with (2.1), (2.4), (2.6) we further define

$$\mathbb{X} = (X_0, X_1, \ldots)$$

$$\mathbb{X}_n = (X_n/\sqrt{EX_n^2}\,,\ X_{n+1}/\sqrt{EX_{n+1}^2}\,,\ \ldots)$$

$$\mathbb{Y} = \sqrt{1 - a^2}\,(Y_0, Y_1, \ldots)$$

$$\mathbb{W} = (W_0, W_1, \ldots).$$

Our discussion will focus on the relationship of the ZCR of $\mathbb{X}$ to that of $\mathbb{X}_n$, $\mathbb{Y}$ and $\mathbb{W}$. The idea is to show that the ZCR in $\mathbb{X}_n$ converges to that of $\mathbb{Y}$ for $|\alpha| < 1$ but it converges to that of $\mathbb{W}$ for $|\alpha| \geq 1$, and then show that the asymptotic ZCR in $\mathbb{X}_n$ and $\mathbb{X}$ are the same. Our first step in this direction is to observe that Lemma 2.1 implies

Lemma 2.2. Assume that $\{\varepsilon_t\}$ is Gaussian white noise and that $X_0 \equiv \xi$ is a zero-mean normal random variable independent of $\{\varepsilon_1,\varepsilon_2,\ldots\}\varepsilon$. Then as $n \to \infty$ we have

(i) If $|\alpha| < 1$ then $\mathbb{X}_n$ converges to $\mathbb{Y}$ in distribution uniformly.

(ii) If $|\alpha| > 1$ then $\mathbb{X}_n$ converges to $\mathbb{W}$ in distribution uniformly.

(iii) If $|\alpha| = 1$ then $\mathbb{X}_n$ converges to $\mathbb{W}$ in distribution.

Let $\mathcal{D}_N(\mathbb{Y})$, $\mathcal{D}_N(\mathbb{W})$ be the numbers of zero-crossings in $(Y_0,Y_1,\ldots,Y_N)$ and $(W_0,W_1,\ldots,W_N)$ respectively. Because $\mathbb{Y}$ is ergodic (1.2) and (1.3) imply that

$$\lim_{N\to\infty} \frac{1}{N}\mathcal{D}_N(\mathbb{Y}) = \frac{1}{\pi}\cos^{-1}\alpha \quad \text{a.s.}$$

Also, we clearly have

$$\frac{1}{N}\mathcal{D}_N(\mathbb{W}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha}{|\alpha|}\right) \quad \text{a.s.}$$

Therefore by bounded convergence

$$\underset{N\to\infty}{\mathrm{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbb{Y}) = \frac{1}{\pi}\cos^{-1}\alpha \tag{2.7}$$

and

$$\underset{N\to\infty}{\mathrm{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbb{W}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha}{|\alpha|}\right). \tag{2.8}$$

We will show that (2.7), (2.8) also hold true for $\mathbb{X}$ depending on whether $|\alpha| < 1$ or $|\alpha| \geq 1$. In passing we note that (2.7) was also established in [1, ch. 5] by a different method whereby $\mathbb{Y}$ is approximated by an M-dependent Gaussian process.

Consider first the case $|\alpha| = 1$ and let $M \geq 2$ be fixed. Define by

$$\mathcal{D}_M(\mathbb{X}_n) = \sum_{j=1}^{M} (X_{n+j} - X_{n+j-1})^2, \quad n \geq 1,$$

the number of zero-crossings in $(X_n, X_{n+1}, \ldots, X_{n+M})$ which is the same as this number for $(X_n/\sqrt{EX_n^2}, \ldots, X_{n+M}/\sqrt{EX_{n+M}^2})$. Then by Lemma 2.2 $\mathbb{X}_n$ converges in distribution to $\mathbb{W}$ as $n \to \infty$ and for all $M \geq 2$

$$\frac{1}{M}\mathcal{D}_M(\mathbb{X}_n) \xrightarrow{p} \frac{1}{\pi}\cos^{-1}\alpha, \quad n \to \infty,$$

and hence, by boundedness,

$$\frac{1}{M}E\mathcal{D}_M(\mathbb{X}_n) \to \frac{1}{\pi}\cos^{-1}\alpha, \quad n \to \infty,$$

and so

$$\lim_{N\to\infty}\frac{1}{N}E\mathcal{D}_N(\mathbb{X}) = \frac{1}{\pi}\cos^{-1}\alpha$$

where $\mathcal{D}_N(\mathbb{X})$ is the number of zero-crossings in $(X_0, X_1, \ldots, X_N)$. Now, since $\frac{1}{\pi}\cos^{-1}\alpha = 0$ or 1 we obtain

$$E\left(\frac{1}{N}\mathcal{D}_N(\mathbb{X}) - \frac{1}{\pi}\cos^{-1}\alpha\right)^2 \leq \frac{1}{N}E\mathcal{D}_N(\mathbb{X}) - \frac{2}{\pi}\cos^{-1}\alpha\,\frac{1}{N}E\mathcal{D}_N(\mathbb{X}) + \frac{1}{\pi}\cos^{-1}\alpha$$

$$\to 0, \quad N \to \infty.$$

Thus

$$\underset{N\to\infty}{\text{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbb{X}) = \frac{1}{\pi}\cos^{-1}\alpha, \quad |\alpha| = 1.$$

Next consider the case $|\alpha| > 1$. Then again from Lemma 2.2 and (1.2)

$$\lim_{N\to\infty}\frac{1}{N}E\mathcal{D}_N(\mathbb{X}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha}{|\alpha|}\right). \tag{2.9}$$

But since $\frac{1}{\pi}\cos^{-1}(\frac{\alpha}{|\alpha|}) = 0$ or 1 again we obtain

$$\underset{N\to\infty}{\mathrm{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbf{X}) = \frac{1}{\pi}\cos^{-1}(\frac{\alpha}{|\alpha|}). \tag{2.10}$$

For the case $|\alpha| < 1$ we have similarly

$$\lim_{N\to\infty}\frac{1}{N}E\mathcal{D}_N(\mathbf{X}) = \frac{1}{\pi}\cos^{-1}\alpha. \tag{2.11}$$

Here, however, $\frac{1}{\pi}\cos^{-1}\alpha$ is neither 0 nor 1 and we cannot use the method employed above. To prove the mean square convergence of the ZCR in this case we need the following additonal lemma.

<u>Lemma 2.3</u>. Let c be a positive constant and let $\mathbf{Y} = (Y_0, Y_1, \ldots)$ be a zero mean stationary Gaussian process with correlation function ρ_k. Assume that $\mathbf{X} = (X_0, X_1, \ldots)$ is a Gaussian process with mean zero and correlation function $\rho_k(t)$ which converges to ρ_k as $t \to \infty$ uniformly in k. Then,

$$\underset{N\to\infty}{\mathrm{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbf{Y}) = c$$

implies

$$\underset{N\to\infty}{\mathrm{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbf{X}) = c\ . \tag{2.12}$$

<u>Proof</u>. Let $\chi_i = \mathbf{1}_{[X_i \geq 0]}$, $\nu_i = \mathbf{1}_{[Y_i \geq 0]}$, $i = 1,2,\ldots,$ be the sequences of indicators obtained by clipping $\mathbf{X}$, $\mathbf{Y}$. Define

$$f(x) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}x$$

$$g(x,y,z) = \frac{1}{8} + \frac{1}{4\pi}(\sin^{-1}x + \sin^{-1}y + \sin^{-1}z).$$

Then f,g are uniformly continuous and for $v > u > t > s$, we obtain from [1, p. 34]

$$EX_sX_t = f(\rho_{t-s}(s))$$

$$EX_sX_tX_u = g(\rho_{t-s}(s),\rho_{u-s}(s),\rho_{u-t}(t)).$$

Also, it is known that there exists a function h such that

$$EX_sX_tX_uX_v = h(\rho_{t-s}(s),\rho_{u-s}(s),\rho_{v-s}(s),\rho_{u-t}(t),\rho_{v-t}(t),\rho_{v-u}(u)).$$

Obviously h is continuous on the compact set

$$\{(x_1,\dots,x_6): \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & 1 & x_4 & x_5 \\ x_2 & x_4 & 1 & x_6 \\ x_3 & x_5 & x_6 & 1 \end{pmatrix} \geq 0, \quad |x_i| \leq 1\}$$

and is therefore uniformly continuous there. Thus by uniform continuity for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(x_1) - f(y_1)| < \varepsilon$$

$$|g(x_1,x_2) - g(y_1,y_2)| < \varepsilon$$

$$|h(x_1,\dots,x_6) - h(y_1,\dots,y_6)| < \varepsilon$$

whenever $|x_i - y_i| < \delta$, $i = 1,\dots,6$. Also, since $\rho_k(n)$ converges to ρ_k uniformly in $k \geq 1$ as $n \to \infty$, we can find N_ε such that

$$|\rho_k(n) - \rho_k| < \delta \quad \text{for all} \quad k \geq 1$$

whenever $n \geq N_\varepsilon$. We therefore have

$$
\begin{aligned}
&|E(\tfrac{1}{N}\mathcal{D}_N(\mathbf{X}))^2 - E(\tfrac{1}{N}\mathcal{D}_N(\mathbf{Y}))^2| \\
&= \frac{1}{N^2}|E(\sum_1^{N_\varepsilon} + \sum_{N_\varepsilon+1}^{N} (x_i - x_{i-1})^2)^2 - E(\sum_1^{N_\varepsilon} + \sum_{N_\varepsilon+1}^{N} (\nu_i - \nu_{i-1})^2)^2| \\
&\le \frac{1}{N^2}|E(\sum_1^{N_\varepsilon} (x_i - x_{i-1})^2)^2 - E(\sum_1^{N_\varepsilon} (\nu_i - \nu_{i-1})^2)^2| \\
&\quad + \frac{2}{N^2}|E \sum_1^{N_\varepsilon}(x_i - x_{i-1})^2 \sum_{N_\varepsilon+1}^{N}(x_j - x_{j-1})^2 - E \sum_1^{N_\varepsilon}(\nu_i - \nu_{i-1})^2 \sum_{N_\varepsilon+1}^{N}(\nu_j - \nu_{j-1})^2| \\
&\quad + \frac{1}{N^2}|E \sum_{N_\varepsilon+1}^{N} (x_i - x_{i-1})^2 - E \sum_{N_\varepsilon+1}^{N} (\nu_i - \nu_{i-1})^2| \\
&\le \frac{N_\varepsilon^2 + 2N_\varepsilon(N - N_\varepsilon)}{N^2} + \frac{16\varepsilon(N - N_\varepsilon)^2}{N^2} .
\end{aligned}
$$

In the above bounding we used the fact $|a - b| \le a \vee b$, for $a,b \ge 0$. Since ε is arbitrary and since N_ε is fixed the last two expressions can be made arbitrarily small as $N \to \varepsilon$, and we proved that

$$\lim_{N\to\infty} \{E(\tfrac{1}{N}\mathcal{D}_N(\mathbf{X}))^2 - E(\tfrac{1}{N}\mathcal{D}_N(\mathbf{Y}))^2\} = 0.$$

Next, by the stationarity of $\mathbf{Y}$ and by hypothesis,

$$\frac{1}{N} E\mathcal{D}_N(\mathbf{Y}) = c ,$$

and so, by the uniform convergence of $\rho_k(t)$ we also have

$$\frac{1}{N} E\mathcal{D}_N(\mathbf{X}) \to c , \qquad N \to \infty,$$

and (2.12) is proved. □

Denote by $\mathbf{X}_-$ the process X_t in (2.1) for $t = -1,-2,\ldots$. That is,

$$\mathbf{X}_- = (X_{-1}, X_{-2}, \ldots).$$

Also, recall the stationary solution (2.5) $\{Z_t\} \equiv \mathbf{Z}$. We can study the ZCR in $\mathbf{X}_-$ by comparing it to that of $\mathbf{Z}$ following the above procedure. In this case $\mathbf{X}_-$ plays the role of $\mathbf{X}$ while $\mathbf{Z}$ plays the role of $\mathbf{Y}$. Let $\mathcal{D}_N(\mathbf{X}_-)$ be the number of zero-crossings in $X_{-1}, X_{-2}, \ldots, X_{-N-1}$, or

$$\mathcal{D}_N(\mathbf{X}_-) = \sum_{j=1}^{N} (x_{-j} - x_{-j-1})^2.$$

We can prove the convergence of $\mathcal{D}_N(\mathbf{X}_-)/N$ in precisely the same way used to prove the convergence of $\mathcal{D}_N(\mathbf{X})/N$. Collecting the above results we have

<u>Theorem 2.1</u>. Assume that $\{X_t\}$ follows the first order autoregressive model (2.1) where $\{\varepsilon_t\}$ is Gaussian white noise and where $X_0 \equiv \xi$ is a zero mean normal random variable independent of $\{\varepsilon_t\}$. Then

$$\underset{N\to\infty}{\text{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathbf{X}) = \begin{cases} \frac{1}{\pi}\cos^{-1}\alpha, & |\alpha| < 1 \\ \frac{1}{\pi}\cos^{-1}(\frac{\alpha}{|\alpha|}), & |\alpha| \geq 1 \end{cases}$$

$$\underset{N\to\infty}{\text{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathbf{X}_-) = \begin{cases} \frac{1}{\pi}\cos^{-1}\frac{1}{\alpha}, & |\alpha| > 1 \\ \frac{1}{\pi}\cos^{-1}(\frac{\alpha}{|\alpha|}), & |\alpha| \leq 1 . \end{cases}$$

Evidently, the asymptotic ZCR depends on α only and is independent of the magnitude of $\{\varepsilon_t\}$. Figure 2.1 portrays the convergence of the ZCR for various values of α. For $\alpha \geq 1$ the path of the ZCR is much smoother than the paths corresponding to $|\alpha| < 1$. Our observation has been that this path property sets apart the "stationary" from the "nonstationary" cases.

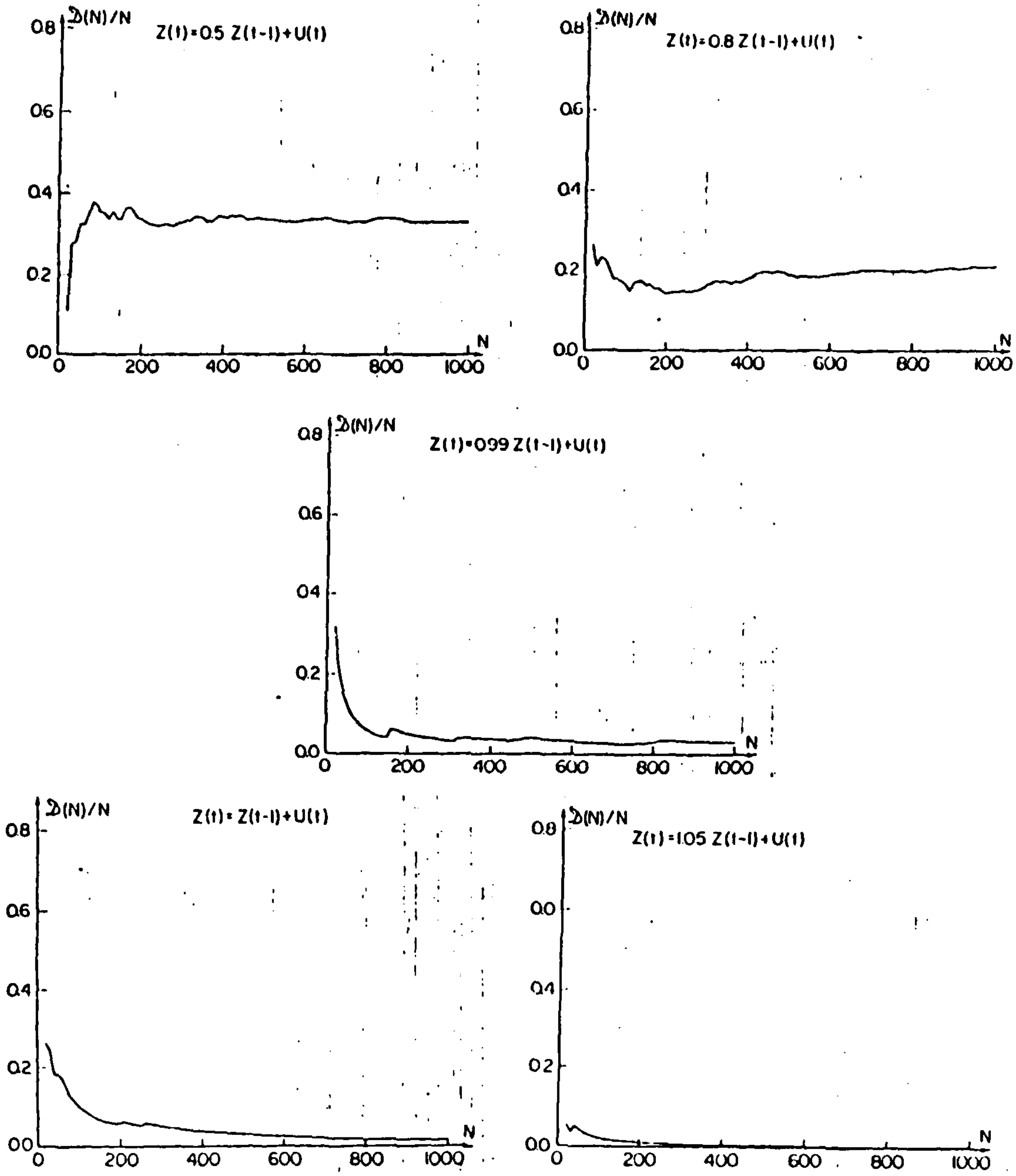


Figure 2.1. ZCR paths of AR(1) processes with $\alpha = (0.5, 0.8, 0.99, 1.0, 1.05)$. Convergence is towards $(0.333, 0.205, 0.045, 0, 0)$ respectively. $\sigma_\varepsilon^2 = 1$. $(\mathcal{D}(N)/N \equiv \mathcal{D}_N(\mathbf{X})/N)$

3. The ZCR of the general AR(2) process

Consider the second order autoregressive process $\{X_t\}$ defined by the stochastic difference equation

$$(1-\alpha B)(1-\beta B)X_t = \varepsilon_t, \quad t = 0,\pm 1,\ldots \tag{3.1}$$

where $\{\varepsilon_t\}$ is Gaussian white noise with variance σ_ε^2, and where $X_{-1} = \xi$, $X_{-2} = \zeta$ are two initial values independent of $\{\varepsilon_t\}$. As before, B is the backward shift operator. Clearly, $\{X_t\}$ is not stationary but the equation (3.1) has a stationary solution which depends on α,β. In what follows, we will show that the asymptotic ZCR of $\{X_t\}$ is determined by α,β but is independent of the magnitude of $\{\varepsilon_t\}$. The method of proof is identical to that given in the first order case. We start off by determining the asymptotic correlations of $\{X_t\}$.

The general solution of (3.1) for $t \geq 0$ is given by

$$\begin{aligned}
X_t &= A_1\alpha^{t+2} + B_1\beta^{t+2} + \frac{1}{\alpha-\beta}\sum_{j=0}^{t}(\alpha^{j+1}-\beta^{j+1})\varepsilon_{t-j}, \quad \alpha \neq \beta, \quad \alpha,\beta \in R^1 \\
&= \alpha^{t+2}(A_2 + B_2(t+2)) + \sum_{j=0}^{t}\alpha^j(j+1)\varepsilon_{t-j}, \quad \alpha = \beta \in R^1 \\
&= \rho^{t+2}(A_3\cos(t+2)\theta + B_3\sin(t+2)\theta) + \sum_{j=0}^{t}\frac{\sin(j+1)\theta}{\sin\theta}\rho^j\varepsilon_{t-j}, \\
&\qquad \alpha = \bar{\beta} = \rho e^{i\theta}, \quad \rho > 0
\end{aligned}$$

where

$$A_1 = \frac{1}{\alpha-\beta}(\zeta - \beta\xi) \qquad B_1 = \frac{1}{\alpha-\beta}(\alpha\xi - \zeta)$$

$$A_2 = \zeta \qquad B_2 = \frac{1}{\alpha}\xi - \zeta$$

$$A_3 = \zeta \qquad B_3 = \frac{1}{\sin\theta}\left(\frac{1}{\rho}\xi - \zeta\sin\theta\right).$$

Note that the case $\alpha = \bar{\beta} = e^{i\theta}$ corresponds to the unit roots $e^{i\theta}$, $e^{-i\theta}$. Under the assumption that ξ, ζ have zero means and finite variances, we have

(i) $\alpha \neq \beta, \ \alpha, \beta \in R^1$

$$EX_t^2 = EA_1^2\alpha^{2t+4} + EB_1^2\beta^{2t+4} + E(A_1B_1)(\alpha\beta)^{t+2}$$
$$+ \frac{\sigma_\varepsilon^2}{(\alpha-\beta)^2}\left\{\frac{1-\alpha^{2t+2}}{1-\alpha^2}\,\alpha^2 + \frac{1-\beta^{2t+2}}{1-\beta^2}\,\beta^2 - 2\,\frac{1-(\alpha\beta)^{t+1}}{1-\alpha\beta}\,\alpha\beta\right\} \qquad (3.2)$$

$$EX_tX_{t+k} = EA_1^2\alpha^{2t+4+k} + EB_1^2\beta^{2t+4+k} + E(A_1B_1)(\alpha\beta)^{t+2}(\alpha^k+\beta^k)$$
$$+ \frac{\sigma_\varepsilon^2}{(\alpha-\beta)^2}\left\{\alpha^{2+k}\,\frac{1-\alpha^{2t+2}}{1-\alpha^2} + \beta^{2+k}\,\frac{1-\beta^{2t+2}}{1-\beta^2} - \alpha\beta\,\frac{1-(\alpha\beta)^{t+1}}{1-\alpha\beta}(\alpha^k+\beta^k)\right\} \qquad (3.3)$$

where if $\alpha = 1$ we define $(1-\alpha^{t+1})/(1-\alpha) \equiv t+1$.

(ii) $\alpha = \beta \in R^1$

$$EX_t^2 = \alpha^{2t+4}\{EA_2^2 + 2EA_2B_2(t+2) + EB_2^2(t+2)^2\} + \sigma_\varepsilon^2\sum_{j=0}^{t}\alpha^{2j}(j+1)^2 \qquad (3.4)$$

$$EX_tX_{t+k} = \alpha^{2t+4+k}\{EA_2^2 + E(A_2B_2)(2t+4+k) + EB_2^2((t+2)(t+2+k))\}$$
$$+ \sigma_\varepsilon^2\sum_{j=0}^{t}\alpha^{2j+k}(j+1)(j+1+k). \qquad (3.5)$$

When $\alpha > 1$, in the limit as $t \to \infty$ of the ratio of (3.5) to (3.4) we use the facts that

$$\lim_{t\to\infty}\frac{1}{\alpha^{2t}(t+2)^2}\sum_{j=0}^{t}\alpha^{2j}(j+1)^2 = \frac{\alpha^2}{\alpha^2-1}$$

and

$$\lim_{t\to\infty}\frac{1}{\alpha^{2t}(t+2)^2}\sum_{j=1}^{t}\alpha^{2j} = 0\,.$$

(iii) $\alpha = \bar{\beta} = \rho e^{i\theta}$, $\theta \in (0,\pi)$, $\rho > 0$

$$EX_t^2 = \rho^{2t+4}\{EA_3^2\cos^2(t+2)\theta + EB_3^2\sin^2(t+2)\theta + EA_3B_3\sin(2t+4)\theta\} + \frac{\sigma_\varepsilon^2}{\sin^2\theta}\sum_{j=0}^{t}\sin^2(j+1)\theta\,\rho^{2j} \tag{3.6}$$

$$\begin{aligned} EX_tX_{t+k} = \rho^{2t+4+k}\{&EA_3^2\cos(t+2)\theta\cos(t+2+k)\theta + EB_3^2\sin(t+2)\theta\sin(t+2+k)\theta \\ &+ EA_3B_3\sin(2t+4+k)\theta\} \\ &+ \sigma_\varepsilon^2\sum_{j=0}^{t}\frac{\sin(j+1)\theta\sin(j+1+k)\theta}{\sin^2\theta}\rho^{2j+k} \end{aligned} \tag{3.7}$$

Observe that

$$\sum_{j=0}^{\infty}\sin^2(j+1)\theta = \infty$$

and

$$\sum_{j=0}^{t}\sin(j+1)\theta\sin(j+1+k)\theta$$

$$= \sum_{j=0}^{t}\sin^2(j+1)\theta\cos k\theta + \frac{\sin 2\theta + \sin(2t+2)\theta - \sin(2t+4)\theta - 1}{4(1-\cos\theta)}\sin k\theta$$

are facts used in the limit ratio of (3.7) to (3.6).

Forming the appropriate ratios of the above expressions and passing to the limit as $t \to \infty$ we obtain the asymptotic correlation as follows.

Lemma 3.1. Assume $\{X_t\}$ follows (3.1), $\{\varepsilon_t\}$ is white noise independent of $X_{-1} \equiv \xi$, $X_{-2} \equiv \zeta$, $E\xi = E\zeta = 0$, $E\xi^2, E\zeta^2 < \infty$. Then for $s \geq 1$

(i) $\alpha \neq \beta, \quad \alpha,\beta \in R^1$

$$\lim_{t\to\infty} \rho_s(t) = \frac{\alpha^{s+1}(1-\beta^2) - \beta^{s+1}(1-\alpha^2)}{\alpha(1-\beta^2) - \beta(1-\alpha^2)}, \quad |\alpha| < 1, \quad |\beta| < 1, \text{ uniformly in } s \geq 1$$

$$= (\frac{\beta}{|\beta|})^s, \quad |\beta| > |\alpha| \geq 1 \quad \text{or} \quad |\beta| \geq 1, \quad |\alpha| < 1$$

$$= (\frac{\alpha}{|\alpha|})^s, \quad |\alpha| > |\beta| \geq 1 \quad \text{or} \quad |\alpha| \geq 1, \quad |\beta| < 1.$$

(ii) $\alpha = \beta \in R^1$

$$\lim_{t\to\infty} \rho_s(t) = \alpha^s(1 + \frac{s(1-\alpha^2)}{1+\alpha^2}), \quad |\alpha| < 1, \text{ uniformly in } s \geq 1$$

$$= (\frac{\alpha}{|\alpha|})^s, \quad |\alpha| \geq 1.$$

(iii) $\alpha = \overline{\beta} = \rho e^{i\theta}, \quad \theta \in (0,\pi), \quad \rho > 0$

$$\lim_{t\to\infty} \rho_s(t) = \rho^s \cos(s\theta) + \frac{(1-\rho^2)\cos\theta \sin s\theta}{(1+\rho^2)\sin\theta}\rho^s, \quad \rho < 1, \text{ uniformly in } s \geq 1$$

$$= \cos(s\theta), \quad \rho = 1, \text{ uniformly in } s \geq 1.$$

Proof. The proof is straightforward and is similar to the one given in Lemma 2.1. □

The stationary solution of (3.1) in terms of a realizable moving average is given by

$$Y_t = \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha-\beta} \varepsilon_{t-k}, \quad \alpha \neq \beta, \quad \alpha,\beta \in R^1, \quad |\alpha| < 1, \quad |\beta| < 1$$

$$= \sum_{k=0}^{\infty} (k+1)\alpha^k \varepsilon_{t-k}, \quad \alpha = \beta \in R^1, \quad |\alpha| < 1$$

$$= \sum_{k=0}^{\infty} \frac{\sin(k+1)\theta}{\sin\theta} \rho^k \varepsilon_{t-k}, \quad \alpha = \overline{\beta} = \rho e^{i\theta}, \quad 0 < \rho < 1,$$

$t = 0,\pm1,\ldots$. Define

$$Z_k = A\cos k\theta + B\sin k\theta, \quad k=0,1,2,\ldots ,$$

where A,B are independent N(0,1) random variables and

$$W_k = \left(\frac{\alpha \vee \beta}{|\alpha \vee \beta|}\right)^k X, \qquad X \sim N(0,1),$$

$k = 0,1,\ldots,$ and let

$$\mathbb{Y} = \frac{1}{\sqrt{EY_t^2}}(Y_0, Y_1, \ldots)$$

$$\mathbb{Z} = (Z_0, Z_1, \ldots)$$

$$\mathbb{W} = (W_0, W_1, \ldots)$$

$$\mathbb{X} = (X_0, X_1, \ldots)$$

$$\mathbb{X}_n = (X_n/\sqrt{EX_n^2}\ ,\ X_{n+1}/\sqrt{EX_{n+1}^2}\ ,\ \ldots)$$

where $\{X_n\}$ is the process in Lemma 3.1. The $\mathbb{Z}$ process is stationary and Gaussian with zero mean and correlation function $\cos(k\theta)$, $k = 0,1,\ldots$. Lemma 3.1 now entails under the Gaussian assumption

<u>Lemma 3.2</u>. Assume $\{X_t\}$ follows the AR(2) model (3.1) where $X_{-1} \equiv \xi$, $X_{-2} \equiv \zeta$ are two normal random variables with zero means and finite variances independent of the Gaussian white noise $\{\varepsilon_t\}$. Then as $n \to \infty$

(i) If $|\alpha| \vee |\beta| < 1$, $\mathbb{X}_n$ converges to $\mathbb{Y}$ in distribution uniformly.

(ii) If $\alpha,\beta \in R^1$, $|\alpha| \vee |\beta| \geq 1$, $\mathbb{X}_n$ converges in distribution to $\mathbb{W}$.

(iii) If $\alpha = \bar{\beta} = e^{i\theta}$, $\mathbb{X}_n$ converges to $\mathbb{Z}$ in distribution uniformly.

As in the first order case, the asymptotic ZCR of $\mathbb{X}$ is the same as that of $\mathbb{X}_n$ but that of $\mathbb{X}_n$ is the same as the asymptotic ZCR of $\mathbb{Y}$ or that of $\mathbb{Z}$ or that of $\mathbb{W}$ depending on α,β. Thus, we need to obtain the limit ZCR in the latter cases.

Under the Gaussian assumption $\mathbb{Y}$ is ergodic. Also, for $|\alpha| < 1$, $|\beta| < 1$, $\alpha,\beta \in R^1$ and $\alpha = \overline{\beta} = e^{i\theta}$, $|\rho| < 1$

$$\frac{EY_t Y_{t+1}}{EY_t^2} = \frac{1}{\pi} \cos^{-1}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right)$$

so that $\mathcal{D}_N(\mathbb{Y})/N \to \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right)$ a.s. as $N \to \infty$. Therefore by bounded convergence also

$$\underset{N\to\infty}{\text{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathbb{Y}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right).$$

Since $\mathbb{Z}$ is a sinusoid its ZCR converges to $\frac{1}{\pi}\cos^{-1}(\cos(\theta)) = \theta/\pi$ a.s. so that again

$$\underset{N\to\infty}{\text{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathbb{Z}) = \frac{\theta}{\pi}.$$

Similarly, it is easy to see that

$$\underset{N\to\infty}{\text{l.i.m}} \frac{1}{N}\mathcal{D}_N(\mathbb{W}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha \vee \beta}{|\alpha \vee \beta|}\right).$$

The convergence of $\mathcal{D}_N(\mathbb{X})/N$ for the cases corresponding to $|\alpha| < 1$, $|\beta| < 1$, $\alpha,\beta \in R^1$, and $\alpha = \overline{\beta} = e^{i\theta}$, can be established by Lemmas 2.3, 3.1, 3.2. The case $\alpha,\beta \in R^1$ and $|\alpha| \vee |\beta| \geq 1$ is much simpler and follows from Lemma 3.1 which gives

$$\lim_{t\to\infty} \rho_1(t) = \frac{\alpha \vee \beta}{|\alpha \vee \beta|}$$

so that

$$\lim_{N\to\infty} \frac{1}{N} E\mathcal{D}_N(\mathbb{X}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha \vee \beta}{|\alpha \vee \beta|}\right).$$

But $\frac{1}{\pi}\cos^{-1}\left(\frac{\alpha \vee \beta}{|\alpha \vee \beta|}\right) = 0$ or 1 and this implies

$$\underset{N\to\infty}{\text{l.i.m}}\ \frac{1}{\pi}\mathcal{D}_N(\mathbf{X}) = \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha \vee \beta}{|\alpha \vee \beta|}\right).$$

The same method can be used to obtain the asymptotic ZCR in $\mathbf{X}_- \equiv (X_{-1}, X_{-2}, \ldots)$ with the appropriate conditions on α, β. This result and the summary of the above discussion are given in

<u>Theorem 3.1</u>. Assume that $\{X_t\}$ follows the AR(2) model (3.1) with initial values $X_{-1} = \xi$, $X_{-2} = \zeta$ that are normal random variables with zero means and finite variances and independent of the Gaussian white noise $\{\varepsilon_t\}$. Then regardless of the magnitude of ε_t

$$\underset{N\to\infty}{\text{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbf{X}) = \begin{cases} \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right), & \alpha,\beta \in R^1, \quad |\alpha| < 1, \quad |\beta| < 1 \\ \frac{1}{\pi}\cos^{-1}\left(\frac{2\rho\cos\theta}{1+\rho^2}\right), & \alpha = \bar{\beta} = \rho e^{i\theta}, \quad |\rho| < 1, \quad 0 \le \theta \le \pi \\ \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha \vee \beta}{|\alpha \vee \beta|}\right), & \alpha,\beta \in R^1, \quad |\alpha| \vee |\beta| \ge 1 \\ \frac{\theta}{\pi}, & \alpha = \bar{\beta} = e^{i\theta}, \quad 0 \le \theta \le \pi . \end{cases}$$

$$\underset{N\to\infty}{\text{l.i.m}}\ \frac{1}{N}\mathcal{D}_N(\mathbf{X}) = \begin{cases} \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right), & \alpha,\beta \in R^1, \quad |\alpha| > 1, \quad |\beta| > 1 \\ \frac{1}{\pi}\cos^{-1}\left(\frac{2\rho\cos\theta}{1+\rho^2}\right), & \alpha = \bar{\beta} = \rho e^{i\theta}, \quad |\rho| > 1, \quad 0 \le \theta \le \pi \\ \frac{1}{\pi}\cos^{-1}\left(\frac{\alpha \wedge \beta}{|\alpha \wedge \beta|}\right), & \alpha,\beta \in R^1, \quad |\alpha| \wedge |\beta| \le 1 \\ \frac{\theta}{\pi}, & \alpha = \bar{\beta} = e^{i\theta}, \quad 0 \le \theta \le \pi . \end{cases}$$

Remark. For the case $\alpha = \bar{\beta} = \rho e^{i\theta}$, $\rho > 1$, $\theta \neq 0$, the limit

$$\lim_{t\to\infty} \frac{EX_t X_{t+k}}{\sqrt{EX_t^2 EX_{t+k}^2}} \tag{3.8}$$

need not exist for every k. To see this put $\xi = \zeta = 0$. Then

$$EX_t^2 = 2\rho^2 \mathrm{Re}\left\{\frac{1}{\rho^2 \sin^2\theta}\left[\frac{1-\rho^{2(t+1)}}{1-\rho^2} - \frac{1-\rho^{2(t+1)}e^{i2(t+1)\theta}}{1-\rho^2 e^{i2\theta}}\, e^{i2\theta}\right]\right\}\sigma_\varepsilon^2$$

and

$$EX_t X_{t+k} = 2\rho^{k+2} \mathrm{Re}\left\{\frac{1}{\rho^2 \sin^2\theta}\left[\frac{1-\rho^{2(t+1)}}{1-\rho^2}\, e^{ik\theta} - \frac{1-\rho^{2(t+1)}e^{i2(t+1)\theta}}{1-\rho^2 e^{i2\theta}}\, e^{i(2+k)\theta}\right]\right\}\sigma_\varepsilon^2 .$$

The problem lies in the fact that $e^{i2(t+1)\theta}$ does not have a unique limit as $t \to \infty$ and that this term cannot be eliminated in the corresponding ratios. More precisely, let

$$a = \frac{1}{\rho^2 - 1}, \qquad be^{i\alpha} = \frac{1}{1-\rho^2 e^{i2\theta}} .$$

Then $a > b$. Take a subsequence $\{t_s\}$ of $\{1,2,\dots\}$ such that

$$\lim_{s\to\infty} e^{it_s\theta} = e^{iu}, \qquad u \in (0,2\pi] .$$

Then

$$\lim_{t\to\infty} \frac{EX_{t_s} X_{t_s+k}}{\sqrt{EX_{t_s}^2\, EX_{t_s+k}^2}} = \frac{a\cos k\theta + b\cos(u+4\theta+k\theta+\alpha)}{\{[a+b\cos(u+4\theta+\alpha)][a+b\cos(u+4\theta+2k\theta+2)]\}^{\frac{1}{2}}} .$$

But this limit depends on u and so 3.8 need not exist for a given k,θ.

4. The case of unit roots

The most interesting case occurs when the equation $(1 - \alpha x)(1 - \beta x) = 0$ has two unit roots $e^{i\theta}$ and $e^{-i\theta}$, $0 \le \theta \le \pi$. In this case (3.1) reduces to

$$X_t = 2 \cos \theta \, X_{t-1} - X_{t-2} + \varepsilon_t, \qquad 0 \le \theta \le \pi. \tag{4.1}$$

By Theorem 3.1 then the ZCR $\mathcal{D}_N(\mathcal{X})/N$ converges to θ/π as $N \to \infty$. Figure 4.1 shows a typical realization of (4.1) for $\theta = \pi/6$. The process oscillates as a sinusoid except that the amplitude tends to increase with t. Because of the sinusoidal oscillation the rate of convergence of the ZCR is very fast and is essentially identical to that observed in a pure sinusoid. In other words, the ZCR of (4.1) is the same whether σ_ε^2 is large, small or even 0. This is illustrated in Table 4.2. The fast rate of convergence is displayed in Figure 4.2 and Tables 4.1, 4.2. Conversely, a fast rate of convergence of the ZCR is a very good indication of the presence of unit roots.

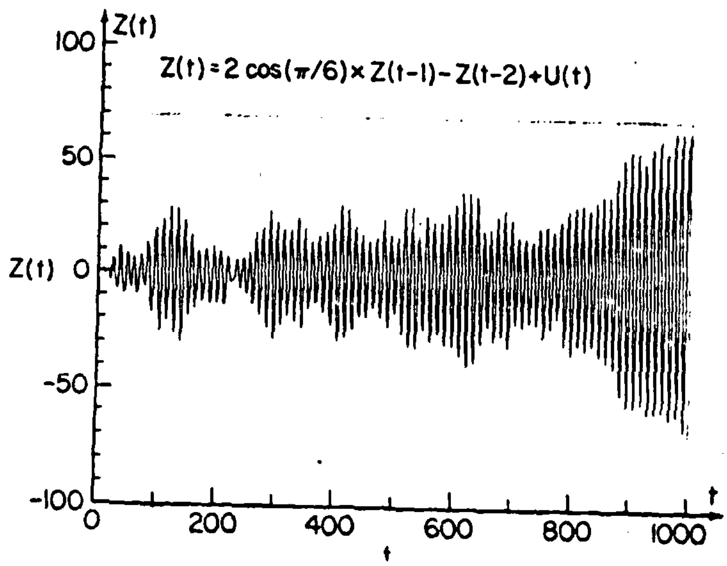
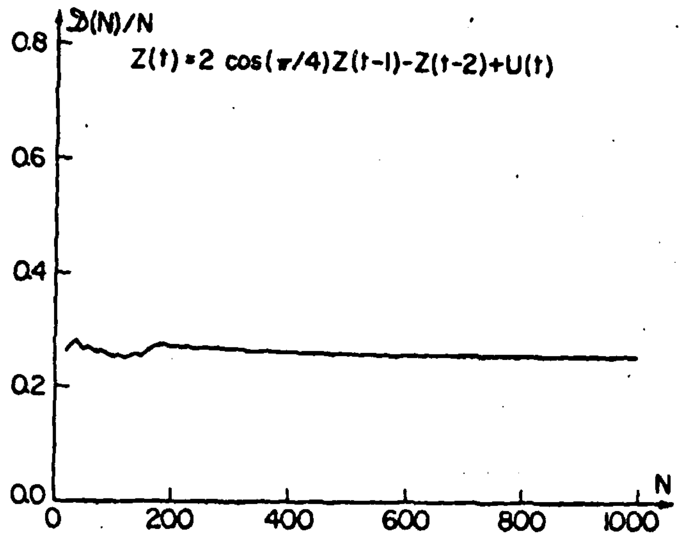


Figure 4.1. Realization of a second order autoregressive process with two unit roots $e^{i\pi/6}$, $e^{-i\pi/6}$, and $\sigma_\varepsilon^2 = 1$.



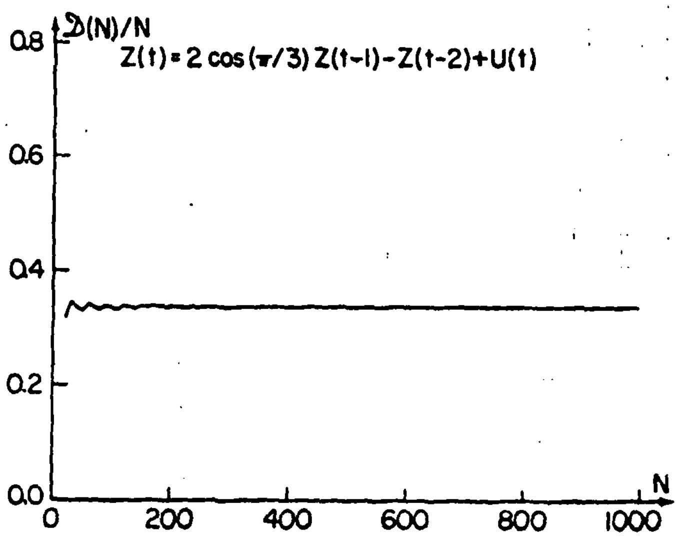


Figure 4.2. ZCR paths of AR(2), $\sigma_\varepsilon^2 = 1$, with unit roots at $e^{\pm i\pi/4}$ and $e^{\pm i\pi/3}$. The convergence is towards 1/4 and 1/3 respectively. $(\mathcal{D}(N)/N \equiv \mathcal{D}_N(\mathcal{X})/N)$

θ/π	1.00000	0.95490	0.50000	0.08134	0.00000
	.90000	.80000	.40000	.10000	.10000
	.99804	.95490	.49608	.08235	.00196
	.99901	.95545	.49802	.08119	.00099
	.99934	.95497	.49868	.08146	.00066
	.99950	.95522	.49900	.08159	.00050
	.99960	.95498	.49920	.08167	.00040
	.99967	.95515	.49934	.08140	.00033
	.99972	.95499	.49943	.08148	.00028
	.99975	.95486	.49950	.08155	.00025
	.99978	.95499	.49956	.08137	.00022
	.99980	.95489	.49960	.08144	.00020
	.99982	.95499	.49964	.08149	.00018
	.99983	.95491	.49967	.08136	.00017
	.99985	.95499	.49969	.08141	.00015
	.99986	.95492	.49971	.08146	.00014
	.99987	.95499	.49973	.08136	.00013
	.99988	.95493	.49975	.08140	.00012
	.99988	.95488	.49976	.08143	.00012
	.99989	.95494	.49978	.08135	.00011
	.99989	.95489	.49979	.08139	.00011
	.99990	.95495	.49980	.08142	.00010
Absolute *Error* N = 10010	0.00010	0.00005	0.00010	0.00007	0.00010

Table 4.1. Convergence to θ/π of $\mathcal{D}_N(\mathbf{Y})/N$, $N = 10, 510, 1010, 1510, 2010, \ldots, 10010$, from the model (4.1) with $\sigma_\varepsilon^2 = 1$. The rate of convergence is fast.

N	θ/π = 0.15915 $\sigma_\varepsilon^2 = 0$	$\sigma_\varepsilon^2 = (8976)^2$	θ/π = 0.79577 $\sigma_\varepsilon^2 = 0$	$\sigma_\varepsilon^2 = (7\cdot 10^6)^2$
10	0.20000	0.20000	0.80000	0.70000
510	0.15882	0.16078	0.79608	0.79412
1010	0.15941	0.15941	0.79505	0.79505
1510	0.15960	0.15894	0.79536	0.79536
2010	0.15920	0.15920	0.79552	0.79552
2510	0.15936	0.15896	0.79562	0.79562
3010	0.15914	0.15880	0.79568	0.79568
3510	0.15926	0.15897	0.79573	0.79573
4010	0.15910	0.15910	0.74576	0.79551
4510	0.15920	0.15898	0.79579	0.79557
5010	0.15928	0.15908	0.79581	0.79561
Absolute Error N = 5010	0.00013	0.00007	0.00003	0.00017
Initial values	$X_1 = 5000.$	$X_2 = -1256.$	$X_1 = 33.$	$X_2 = 55.$

Table 4.2. Convergence to θ/π of $\mathcal{D}_N(\mathbf{X})/N$ from the model (4.1) with σ_ε^2 very large or 0. The convergence is essentially independent of σ_ε^2.

The last remark that unit roots correspond to a fast rate of ZCR convergence can be illustrated by the model (3.1) with $\alpha = \bar{\beta} = \rho e^{i\theta}$, $0 \le \rho \le 1$. In this case (3.1) becomes

$$X_t = 2\rho \cos\theta \, X_{t-1} - \rho^2 X_{t-2} + \varepsilon_t , \tag{4.2}$$

where $0 \le \rho < 1$ means that (4.2) has a stationary solution while $\rho = 1$ corresponds to two unit roots and (4.2) does not have a stationary solution. By Theorem 3.1

$$D_N(\mathcal{X})/N \to \omega \equiv \frac{1}{\pi} \cos^{-1}\left(\frac{2\rho \cos\theta}{1 + \rho^2} \right) . \tag{4.3}$$

Table 4.3 gives the ZCR for $N = 10, 210, 410, 610, \ldots, 5010$, $\rho = 0, 0.5, 1$, and $\theta = 2.5$. From the table it is evident that the rate corresponding to $\rho = 1$ is faster than in the other two cases. (Note that $\rho = 0$ means that X_t is white noise.) Again, as in the first order case, the ZCR path corresponding to the "nonstationary" case is much smoother than the paths produced by the "stationary" series.

$\rho = 0$, $\theta = 2.5$, $\omega = 0.50000$		$\rho = 0.5$, $\theta = 2.5$, $\omega = 0.72144$		$\rho = 1$, $\theta = 2.5$, $\omega = 0.79577$	
ZCR	ERROR	ZCR	ERROR	ZCR	ERROR
.30000	-.20000	.70000	-.02144	.70000	-.09577
.52857	.02857	.74286	.02141	.79048	-.00530
.51463	.01463	.72439	.00295	.79268	-.00309
.51311	.01311	.73115	.00970	.79508	-.00069
.50494	.00494	.72963	.00818	.79506	-.00071
.49109	-.00891	.73960	.01816	.79604	.00026
.48843	-.01157	.73223	.01079	.79587	.00009
.48794	-.01206	.73475	.01331	.79574	-.00003
.49565	-.00435	.73416	.01272	.79565	-.00012
.48453	-.01547	.73425	.01281	.79613	.00036
.48706	-.01294	.73284	.01139	.79602	.00025
.48824	-.01176	.73077	.00932	.79593	.00015
.48963	-.01037	.73320	.01175	.79585	.00008
.48851	-.01149	.73180	.01036	.79579	.00001
.48790	-.01210	.72989	.00845	.79573	-.00005
.48671	-.01329	.72791	.00646	.79568	-.00009
.48692	-.01308	.72368	.00223	.79564	-.00014
.49472	-.00528	.72463	.00319	.79560	-.00017
.49584	-.00416	.72438	.00293	.79584	.00007
.49423	-.00577	.72520	.00375	.79580	.00003
.49377	-.00623	.72369	.00225	.79576	-.00001
.49216	-.00784	.72399	.00255	.79572	-.00005
.49093	-.00907	.72358	.00214	.79569	-.00008
.49176	-.00824	.72256	.00111	.79566	-.00011
.49252	-.00748	.72391	.00246	.79584	.00007
.49441	-.00559	.72236	.00091	.79581	.00003

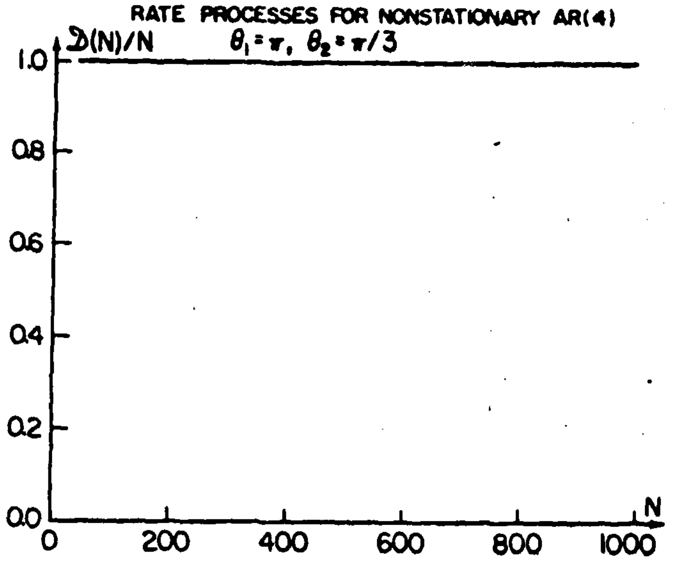
Table 4.3. Convergence of $\mathcal{D}_N(\mathbb{X})/N$, $N = 10, 210, 410, 610, \ldots, 5010$, from 4.2 with $\sigma_\varepsilon^2 = 1$, $X_1 = 10.185$, $X_2 = -5.114$. ERROR $\equiv$ ZCR - ω, ω given in 4.3. The ERROR column indicates the rate of convergence.

4.1. Multiple unit roots and ZCR of differenced series

As noted in the introduction, at present it is not entirely clear how to extend our results to the case of multiple roots. However, we conjecture that the ZCR of the filtered series may determine these roots. An indication of this fact is now outlined.

Let $\nabla \equiv 1 - B$ be the difference operator and consider the ZCR of $\{\nabla^k X_t\}$. Suppose the unit roots are recorded as $e^{i\theta_1}, e^{i\theta_2}, \ldots,$ where $0 \leq \theta_1 < \theta_2 < \ldots \leq \pi$. Then under some conditions, as $k \to \infty$, the ZCR for large N converges to the greatest θ_j/π. We can illustrate this idea by examples. Figure 4.3 displays the ZCR paths obtained by repeated differencing. The lowest path corresponds to no differencing, the one above it corresponds to the ..st difference and so on. This monotonicity is due to repeated differencing which acts as sequential highpass filtering. In the two cases in Figure 4.3 the roots are $e^{\pm i\pi}$, $e^{\pm i\pi/3}$ and $e^{\pm i0.85\pi}$, $e^{\pm i0.15\pi}$, respectively, and the processes are AR(4). In both cases convergence to the greatest θ_j/π occurs as $k \to \infty$. We shall pursue this line of thought elsewhere.

(a)



(b)

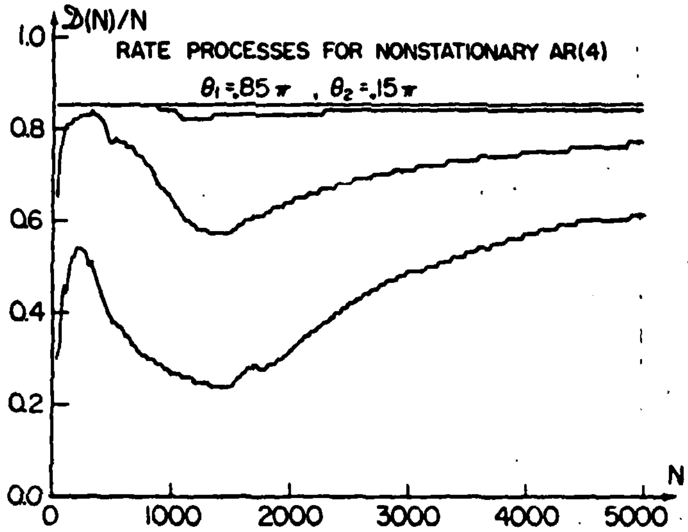


Figure 4.3. ZCR paths from differenced AR(4), $k = 0,1,\ldots,7$, with four unit roots converging to the largest θ_j/π. In (a) all the paths are the same. In (b) the convergence of the paths tc a straight line is fast. $(\mathcal{D}(N)/N \equiv \mathcal{D}_N(\mathbf{X})/N)$

References

[1] Kedem, B. (1980). <u>Binary Time Series</u>, Dekker, New York.